UDC 004.438
doi:10.15217/issn1684-8853.2015.2.2

# THREE-LEVEL CRETAN MATRICES OF ORDER 37

**N. A. Balonin**[a], *Dr. Sc., Tech., Professor, korbendfs@mail.ru*
**Jennifer Seberry**[b], *PhD, Professor of Computer Science, jennifer_seberry@uow.edu.au*
**M. B. Sergeev**[a], *Dr. Sc., Tech., Professor, mbse@mail.ru*
[a]*Saint-Petersburg State University of Aerospace Instrumentation, 67, B. Morskaia St., 190000, Saint-Petersburg, Russian Federation*
[b]*School of Computer Science and Software Engineering, Faculty of Engineering and Information Science, University of Wollongong, NSW, 2522, Australia*

***Purpose:*** *This note discusses three-level orthogonal matrices which were first highlighted by J. J. Sylvester. Hadamard matrices, symmetric conference matrices, and weighing matrices are the best known of these matrices with entries from the unit disk. The goal of this note is to develop a theory of such matrices based on preliminary research results.* ***Methods:*** *Extreme solutions (using the determinant) have been established by minimization of the maximum of the absolute values of the elements of the matrices followed by their subsequent classification.* ***Results:*** *We give a three-level Cretan(37). This is new and the first time such a matrix has been found whose order is other than a $2^k$, k is an even integer. The methods given in this note may be used to construct many more Cretan matrices based on regular Hadamard matrices.* ***Practical relevance:*** *The over-riding aim is to seek Cretan(n) with absolute or relative (local) maximal determinants as they have many applications in image processing and masking. Web addresses are given for other illustrations and other matrices with similar properties. Algorithms to construct Cretan matrices have been implemented in developing software of the research program-complex.*

***Keywords*** *— Hadamard Matrices, Regular Hadamard Matrices, Orthogonal Matrices, Cretan Matrices.*

***AMS Subject Classification:*** *05B20; 20B20.*

## Introduction

In this and further papers we use some names, definitions, notation differently than we have in the past [1]. This, we hope, will cause less confusion, bring our nomenclature closer to common usage and conform for mathematical purists. We have chosen the use of the word level, instead of value for the entries of a matrix, to conform to earlier writings. We note that the strict definition of an orthogonal matrix, $\mathbf{X}$, of order $n$, is that $\mathbf{X}^T\mathbf{X} = \mathbf{X}\mathbf{X}^T = \omega\mathbf{I}_n$, where $\mathbf{I}_n$ is the identity matrix of order $n$. In this paper we consider $\mathbf{S}^T\mathbf{S} = \mathbf{S}\mathbf{S}^T = \omega\mathbf{I}_n$ where $\omega$ is a constant. We call these orthogonal matrices [2, 3]. We refer to [2, 4, 5] for definitions not given below.

## Definitions

**Definition 1.** A *Cretan*($n$) (*CM*) matrix, **S**, is a orthogonal matrix of order $n$ with entries with moduli $\le 1$, where there must be at least one 1 per row and column. The inner product of a row of $CM(n)$ with itself is the *weight* $\omega$. $\mathbf{S}^T\mathbf{S} = \mathbf{S}\mathbf{S}^T = \omega\,\mathbf{I}_n$. The inner product of distinct rows of $CM(n)$ is zero. A μ-level *Cretan*($n$; μ; ω) matrix, $CM(n; \mu; \omega)$, has μ levels or values for its entries.

*Cretan*($n$), or $CM(n)$ orthogonal matrices are studied in [2, 4]. In more general notation these are can be *CM*(*order*), *CM*(*order*; *number of levels* = τ), *CM*(*order*; *number of levels* = τ; *occurrences of levels* = $L_1$, $L_2$, ..., $L_\tau$), *CM*(*order*; *number of levels* = τ; *weight* = ω), *and CM*(*order*; *number of levels* = τ; *weight*; *occurrences of levels in whole matrix*), etc. etc. etc.

The definition of Cretan is not that each variable occurs some number of times per row and column but $L_1$, $L_2$, ..., $L_\tau$ times in the whole matrix. So we have $CM(n; \tau; \omega; L_1, L_2, ..., L_\tau)$ so

$$\begin{pmatrix} -0.5 & 1 & 1 \\ 1 & -0.5 & 1 \\ 1 & 1 & -0.5 \end{pmatrix}$$

is a $CM(3)$, a $CM(3;2)$, a $CM(3;2;2,1)$, a $CM(3;2;2.25)$, a $CM(3;2;2.25;6,3)$ depending on which numbers (in brackets) are currently of interest. We call them Cretan matrices because they were first discussed in this generality at a conference in Crete in July, 2014.

The over-riding aim is to seek $CM(n)$ with absolute or relative (local) maximal determinants as they have many applications in image processing and masking [1, 2].

The matrix *orthogonality equation* $\mathbf{S}^T\mathbf{S} = \mathbf{S}\mathbf{S}^T = \omega\mathbf{I}_n$, is a set of $n^2$ *scalar* equations, giving two kinds of formulae: $g(a, b, s) = \omega$, there are $n$ such equations, and $f(a, b, s) = 0$, there are $n^2 - n$ such equations. We concentrate on two of them: $g(a, b, s) = \omega$, $f(a, b, s) = 0$.

The entries in $\omega\mathbf{I}_n$ which are on the diagonal, are given by the *radius equation* $\omega = g(a, b, s)$, they depend on the choice of $a$, $b$, $s$. At least one of the levels has value 1.

The maximal weight $\omega = n$ arises from Hadamard matrices, and for symmetric conference matrices $\omega = n - 1$. *Orthogonal* matrices can have also irrational values for the weight.

The second equation $f(a, b, s) = 0$ we call the *characteristic equation*, as it allows us to find a formulae for levels $b \le s \le a$.

**Definition 2.** A *regular Hadamard matrix*, $\mathbf{H}$, of order $4t$, has elements plus one and minus one only. It satisfies the *orthogonality equation* $\mathbf{H}^T\mathbf{H} = \mathbf{H}\mathbf{H}^T = 4t\mathbf{I}$. The sum of the entries in any row or column is $2\sqrt{t}$.

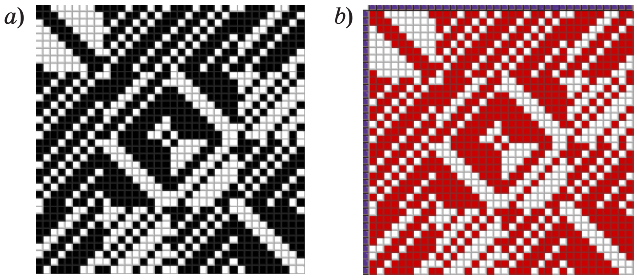
We first note that a computer search undertaken recently in Iran found over 31 million inequivalent regular Hadamard matrices of order 36. We use [5] to give our example, $\mathbf{H}$, for this note.

## A new Cretan(37, 3)

**Construction.** Let $\mathbf{G}(a, b)$ be the 36×36 regular Hadamard matrix [5] with more ones than minus ones in each row and column, and then the ones replaced by "$a$" and minus ones replaced by "$-b$". Let $\mathbf{F}$ be the 37×37 matrix

$$\mathbf{F} = \begin{pmatrix} a & s & \dots & s \\ s & & & \\ \vdots & & \mathbf{G}(a,b) & \\ s & & & \end{pmatrix}.$$

Then if the radius equations are $a^2 + 36s^2 = \omega$, $15a^2 + 21b^2 + s^2 = \omega$; and the characteristic equations are $16a - 21b = 0$, $6a^2 - 18ab + 12b^2 + s^2 = 0$: $\mathbf{F}$ is a Cretan(37;3;27.9388;541,756,72) using the definition (Figure, $a$ and $b$).



■ Orthogonal matrices: the regular Hadamard matrix $\mathbf{H}(36)$ ($a$) and Cretan(37; 3) ($b$)

In the Figure the white square is for element "$a$", black and red (core) square is for element "$-b$", and blue square is for border "$s$".

## Acknowledgements

The authors also wish to sincerely thank Tamara Balonina for converting this paper into printing format. We acknowledge the use of the http://www.mathscinet.ru and http://www.wolframalpha.com sites for the number and symbol calculations in this paper.

## Conclusion

The methods given in this note may be used to construct many more Cretan matrices based on regular Hadamard matrices. The matrices $\mathbf{G}$ and $\mathbf{F}$ both give completely new Cretan matrices $CM(4t, 2)$ and $CM(4t + 1,3)$. They will be studied in depth in paper of J. Seberry, in preparation and add new members to the set of Cretan matrices constructed with one core and one border, observed in [2, 3, 6]. This strongly suggests we have discovered a new branch of Cretan matrices.

## References


1. **Balonin N. A., Mironovsky L. A.** Hadamard Matrices of Odd Order. *Informatsionno-upravliaiushchie sistemy* [Information and Control Systems], 2006, no. 3, pp. 46–50 (In Russian).
2. **Balonin N. A., Sergeev M. B.** Local Maximum Determinant Matrices. *Informatsionno-upravliaiushchie sistemy* [Information and Control Systems], 2014, no. 1(68), pp. 2–15 (In Russian).
3. **Balonin N. A., Seberry Jennifer.** Remarks on Extremal and Maximum Determinant Matrices with Moduli of Real Entries ≤ 1. *Informatsionno-upravliaiushchie sistemy* [Information and Control Systems], 2014, no. 5(72), pp. 2–4.
4. **Hadamard J.** Résolution d'une Question Relative aux Déterminants. *Bulletin des Sciences Mathématiques*, 1893, vol. 17, pp. 240–246 (In French).
5. **Seberry Jennifer.** Regular Hadamard Matrices of Order 36. Available at: http://www.uow.edu.au/jennie/matrices/H36/36R.html (accessed 5 February 2015).
6. **Balonin N. A., Seberry Jennifer.** Two Level Cretan Matrices Constructed via Singer Difference Sets. *Informatsionno-upravliaiushchie sistemy* [Information and Control Systems], 2014, no. 6(73), pp. 2–5.