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CRETAN ($4t$ + 1) MATRICES

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Purpose: *We tried to obtain a Cretan(4t+1) matrix of order 4t+1, i.e. an orthogonal matrix whose elements have moduli ≤1. The only Cretan(4t+1) matrices previously published were of orders 5, 9, 13, 17 and 37.* ***Results:*** *In the paper, we give an infinite number of new Cretan(4t+1) matrices constructed by the use of regular Hadamard matrices, SBIBD(4t+1; k; λ), weighing matrices, generalized Hadamard matrices and Kronecker product. We introduce an inequality for the matrix radius and give a construction for a Cretan matrix of any order n≥5.* ***Practical relevance:*** *Cretan(4t+1) matrices have direct practical applications to the problems of noise-immune coding, compression and masking of video information.*

Keywords *— Hadamard Matrices, Regular Hadamard Matrices, OrthogonalMatrices, Symmetric Balanced Incomplete Block Designs (SBIBD), Cretan Matrices,Weighing Matrices, Generalized Hadamard Matrices, 05B20.*

Introduction

An application in image processing (compression, masking) led to the search for orthogonal matrices, all of whose elements have modulus ≤1 and which have maximal or high determinant.

Cretan matrices were first discussed, per se, during a conference in Crete in 2014. This paper follows closely the joint work of N. A. Balonin, Jennifer Seberry and M. B. Sergeev [1–3].

The orders $4t$ (Hadamard), $4t-1$ (Mersenne), $4t-2$ (Weighing) are discussed in [4–6]. This present work emphasizes the $4t+1$ (Fermat type) orders with real elements ≤1. *Cretan* matrices which are complex, based on the roots of unity or are just required to have at least one 1 are mentioned.

Preliminary Definitions

The absolute value of the determinant of any matrix is not altered by 1) interchanging any two rows, 2) interchanging any two columns, and/or 3) multiplying any row/or column by −1. These equivalence operations are called *Hadamard equivalence operations*. So the absolute value of the determinant of any matrix is not altered by the use of Hadamard equivalence operation.

Write $\mathbf{I}_n$ for the identity matrix of order n, $\mathbf{J}$ for the matrix of all 1's and let ω be a constant. An *orthogonal* matrix, $\mathbf{S}$, of order n, is square, has real entries and satisfies $\mathbf{SS}^{\mathrm{T}} = \omega\mathbf{I}_n$. The *core* of a matrix is formed by removing the first row and column.

A *Cretan* matrix, $\mathbf{S}$, of order n has entries with modulus ≤1 and at least one 1 per row and column. It satisfies $\mathbf{SS}^{\mathrm{T}} = \omega\mathbf{I}_n$ and so it is an orthogonal matrix. A *Cretan*(n; τ; ω) matrix, or *CM*(n; τ; ω) has τ levels or values for its entries [1].

An *Hadamard matrix* of order n has entries ±1 and satisfies $\mathbf{HH}^{\mathrm{T}} = n\mathbf{I}_n$ for $n = 1, 2, 4t$, $t > 0$ an integer. Any Hadamard matrix can be put into *normalized form*, that is having the first row and column all plus 1s using Hadamard equivalence operations: that is it can be written with a core. A *regular Hadamard matrix* of order $4m^2$ has $2m^2 \pm m$ elements 1 and $2m^2 \mp m$ elements −1 in each row and column (see [7, 8]).

Hadamard matrices and weighing matrices are well known orthogonal matrices. We refer to [2, 7–10] for more details and other definitions. The reader is pointed to [11–13] for details of generalized Hadamard matrices, Butson — Hadamard matrices and generalized weighing matrices.

For the purposes of this paper we will consider an *SBIBD*(v, k, λ), $\mathbf{B}$, to be a $v\times v$ matrix, with entries 0 and 1, k ones per row and column, and the inner product of distinct pairs of rows and/or columns λ. This is called the *incidence matrix* of the SBIBD. For these matrices $\lambda(v-1) = k(k-1)$, $\mathbf{BB}^{\mathrm{T}} = (k-\lambda)\mathbf{I} + \lambda\mathbf{J}$ and $\det\mathbf{B} = k\left(k-\lambda\right)^{\frac{v-1}{2}}$.

For every *SBIBD*(v, k, λ) there is a complementary *SBIBD*(v, $v-k$, $v-2k+\lambda$). One can be made from the other by interchanging the 0's of one with the 1's of the other. The usual SBIBD convention that $v > 2k$ and $k > 2\lambda$ is followed.

We now define our important concepts the *orthogonality equation*, the *radius equation*(*s*), the *characteristic equation*(*s*) and the *weight* of our matrices.

Definition 1 (orthogonality equation, radius equation(s), characteristic equation(s), weight). Consider the matrix $\mathbf{S} = (s_{ij})$ of order n comprising the variables $x_1, x_2, \ldots, x_\tau$.

The *matrix orthogonality equation*

$$\mathbf{S}^{\mathrm{T}}\mathbf{S} = \mathbf{SS}^{\mathrm{T}} = \omega\mathbf{I}_n \tag{1}$$

yields two types of equations: the n equations which arise from taking the inner product of each row/column with itself (which leads to the diagonal elements of $\omega\mathbf{I}_n$ being ω) are called *radius equation*(*s*), $g(x_1, x_2, ..., x_\tau) = \omega$, and the $n^2 - n$ equations, $f(x_1, x_2, ..., x_\tau) = 0$, which arise from taking inner products of distinct rows of **S** (which leads to the zero off diagonal elements of $\omega\mathbf{I}_n$ are called *characteristic equation*(*s*)). *Cretan* matrices must satisfy the three equations: the orthogonality equation (1), the radius equation and the characteristic equation(s).

Notation: We use **CM**(n; τ; ω; det(*optional*); ($t_1, t_2, ..., t_\tau$)), or just **CM**(n; τ; ω), where $t_1, t_2, ..., t_\tau$ are the possible values (or levels) of the elements in **CM**.

Inequalities

Some inequalities are known for matrices which have real entries ≤1. Hadamard matrices, $\mathbf{H} = (h_{ij})$, which are orthogonal and with entries ±1 satisfy the equality of Hadamard's inequality (2) [9]

$$\det\left(\mathbf{HH}^{\mathrm{T}}\right) \le \prod_{i=1}^{n}\sum_{j=1}^{n}\left|h_{ij}\right|^2, \tag{2}$$

have determinant $\le n^{\frac{n}{2}}$. Further Barba [14] showed that for matrices, **B**, of order n whose entries are ±1:

$$\det\mathbf{B} \le \sqrt{2n-1}\,(n-1)^{\frac{n-1}{2}}$$

or asymptotically $\approx 0.858(n)^{\frac{n}{2}}$.

For $n = 9$ Barba's inequality gives $\sqrt{17}\times 8^4 =$ $= 16\,888.24$. The Hadamard inequality gives 19 683 for the bound on the determinant of the ±1 matrix of order 9. So the Barba bound is better for odd orders. We thank Professor Christos Koukouvinos for pointing out to us that the literature, see Ehlich and Zeller, [15], yields a ±1 matrix of order 9 with determinant 14 336. These bounds have not been met for $n = 9$.

Koukouvinos also pointed out that in Raghavarao [16] a ±1 matrix of order 13 with determinant $14\,929\,920 \approx 1.49 \times 10^7$ is given. This is the same value given for $n = 13$ given by Barba's inequality. The Hadamard inequality gives 1.74×10^7 for the bound on the determinant of the ±1 matrix of order 13.

These bounds have been significantly improved by Brent and Osborn [17] to give $\le (n+1)^{\frac{(n-1)}{2}}$.

Wojtas [18] showed that for matrices, **B**, whose entries are ±1, of order $n \equiv 2 \pmod 4$ we have

$$\det\mathbf{B} \le 2(n-1)(n-2)^{\frac{n-2}{2}}$$

or asymptotically $\approx 0.736(n)^{\frac{n}{2}}$.

This gives a determinant bound ≤73 728 for order 10 whereas the weighing matrix of order 10 has determinant $9^5 = 59\,049$.

We observe that the determinant of a **CM**(n; τ; ω; det) is always $\omega^{\frac{n}{2}}$.

Hence we can rewrite the known inequalities of this subsection noting that only the Hadamard in equality applies generally for elements with modulus ≤1. Thus we have:

Theorem 1. Hadamard — *Cretan* Inequality. *The radius of a Cretan matrix of order n is ≤n.*

Two Trivial *Cretan(n)* Families

The next two families are included for completeness.

The Basic Family

Lemma 1. *Consider* $\mathbf{C} = a\mathbf{I} + b(\mathbf{J} - \mathbf{I})$ *of order n, a, b variables. This gives a* $\mathbf{CM}\left(n; 2; 1 + \frac{4(n-1)}{(n-2)^2}\right)$ *matrix of order n, i.e. a* $\mathbf{CM}\left(n; 2; 1 + \frac{4(n-1)}{(n-2)^2}; \det; \left(1, \frac{-2}{n-2}\right)\right)$.

Proof. Writing **C** with a on the diagonal and other elements b, the radius and characteristic equations become

$$a^2 + (n-1)b^2 = \omega \text{ and } 2a + (n-2)b = 0.$$

Hence with $a = 1$ and $b = \frac{-2}{n-2}$ we have $\omega = 1 + \frac{4(n-1)}{(n-2)^2}$ for the required **CM**(n) matrix.

Remark 1. For n = 7, 9, 11, 13 this gives $\omega = 1\frac{24}{25}, 1\frac{32}{49}, 1\frac{40}{81}$ and $1\frac{48}{121}$ respectively. These determinants are very small. However they do give a **CM**(n; 2) for all integers $n > 0$.

Known Families

The following results may be found in [19] and [6].

Proposition 1. [*Cretan*($4t$)]. *There is a Cretan*($4t$; 2; $4t$) *for every integer* $4t$ *for which there exists an Hadamard matrix.*

Proposition 2. [*Cretan*($4t - 1$)]. *There are Cretan*($4t - 1$; 2; ω), $\omega = 4t + 1 - \sqrt{t}$ *and* $\omega = \frac{2t^3 + t - 2t(2t-1)\sqrt{t}}{(t-1)^2}$ *for every integer* $4t$ *for which there exists an Hadamard matrix.*

The next two results are easy for the knowledgable reader and merely mentioned here.

Proposition 3. [*Cretan*($4t - 2$)]. *There are Cretan*($4t - 2$; 3; k) *whenever there is a* $\mathbf{W}(4t - 2, k)$

weighing matrix. For $k = 4t - 3$, the sum of two squares, and a $\mathbf{W}(4t-2, 4t-3)$ *is known, the complex Cretan matrix* $\mathbf{CM}(4t-2; 3; 4t-2)$ *has elements* $i = \sqrt{-1}$, 1 *or* -1.

Proposition 4. [*Cretan*(np)]. *There are complex Cretan*(np; p; n), *when ever there exists a generalized Hadamard matrix based on the p th roots of unity.*

The Additive Families

We will illustrate this construction using two *Cretan* matrices to give a *Cretan* matrix whose order is the sum of their orders. This shows how many possible matrices we might find for any n but again all the determinants are small.

Lemma 2. *Let* **A** *and* **B** *be* $\mathbf{CM}(n_1; 3; \omega_1)$ *and* $\mathbf{CM}(n_2; 3; \omega_2)$ *respectively. Then* $\mathbf{A} \oplus \mathbf{B}$ *given by*

$$\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{bmatrix}$$

is a $\mathbf{CM}(n_1 + n_2; 4; \omega)$ *matrix of order* $n_1 + n_2$ *with* $\omega = \min(\omega_1, \omega_2)$. (*Note it does not have one* 1 *per row and column.*)

Remark 2. We note using smaller $\mathbf{CM}(n_i; \tau; \omega_i)$ gives many inequivalent $\mathbf{CM}(n; \tau; \omega)$ for any order $n = \sum_i n_i$ but the elements of all but the smallest sub matrix will not contribute 1 to the resulting *Cretan* matrix.

Now with $n = n_1 + n_2$ for $21 = 4 + 17$, $5 + 16$, $6 + 15$, $7 + 14$, $8 + 13$, $9 + 12$, $10 + 11$ plus other combinations, the sub matrices of orders n_1 and n_2 contribute differently to τ and ω. This means

Proposition 5. *There is a Cretan*(n; τ; ω) *for every integer* n.

In the section on Kronecker product of *Cretan* matrices we explore the same Proposition 5 for more interesting τ.

Constructions for *Cretan*($4t + 1$; τ) Matrices

We now describe a number of constructions for *Cretan*($4t + 1$) matrices.

Constructions using SBIBD

• **2-level *Cretan*($4t + 1$) matrices via *SBIBD*($v = 4t + 1, k, \lambda$)**

The following Theorem is a special case of the construction for 2-level *Cretan*($v = 4t + 1$) given in [6]. It also yields a valid $\mathbf{CM}(37; 2)$.

Theorem 2 [6]. *Let* **S** *be a* $\mathbf{CM}(v = 4t + 1; 2; \omega; (a, b))$ *based on SBIBD*($v = 4t + 1$, k, λ) *then* $a = 1$, $b = \dfrac{(k-\lambda) \pm \sqrt{k-\lambda}}{v - 2k + k}$ *and* $\omega = ka^2 + (v - k)b^2$, *provided* $|b| \le 1$.

Example 1. Using the La Jolla Repository http://www.ccrwest.org/ds.html of difference sets we obtain an *SBIBD*(37, 9, 2). Using Theorem 2 we obtain $\mathbf{CM}(37; 2; 12.325; (1, 0.345))$ and $\mathbf{CM}(37; 2; 9.485; (1, 0.132))$. The complementary *SBIBD*(37, 28, 21) does not give any *Cretan* matrix as $|b|$is ≥ 1.

We especially note the (45, 12, 3) difference set, where the occurrence of the $Cretan\left(45; 2; 20\frac{1}{4}\right)$ matrix and the $Cretan\left(45; 2; 14\frac{1}{16}\right)$ matrices both arise from the *SBIBD*(45, 12, 3): the complementary *SBIBD*(45, 33, 24) does not yield any *Cretan* matrix.

Example 2. Orthogonal matrices of orders 13 and 21 may be constructed by using the *SBIBD*(13, 4, 1) and *SBIBD*(21, 5, 1) given in [20]. $\mathbf{CM}\left(13; 2; 9; 60; \left(1, \frac{3 \pm \sqrt{3}}{6}\right)\right)$ and $\mathbf{CM}\left(21; 2; 10; \left(1, -\frac{1}{6}\right)\right)$ are given in Fig. 1, *a*, *b*.

All the examples of *SBIBD*($4t + 1$, k, λ) that we have given from the La Jolla Repository have been constructed using difference sets. Most of those we give arise from Singer difference sets and finite geometries: these *SBIBD*($(p^{n+1} - 1)/(p - 1)$, $(p^n - 1)/(p - 1)$, $(p^{n-1} - 1)/(p - 1)$) difference sets are denoted as $PG(n, p)$. The bi-quadratic type constructions are due to Marshall Hall [21]. There are many *SBIBD* constructed without using difference sets.

• **Bordered Constructions**

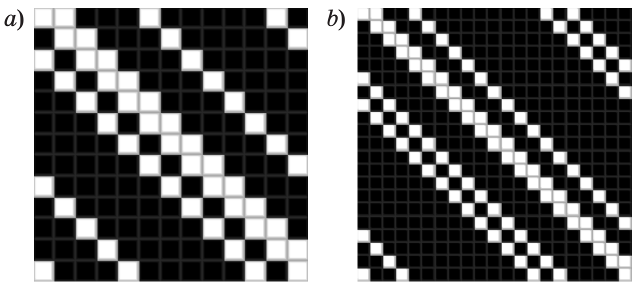
We do not elaborate on the next theorem here but note it gives many *Cretan* matrices $\mathbf{CM}(v + 1)$.

Theorem 3. *The matrix* **C** *below can be used to construct many* $\mathbf{CM}(v + 1; \tau; \omega)$ *with borders by replacing the matrix* **B** *by an SBIBD*(v, k, λ).

When a matrix **C** is written in the following form

$$\mathbf{C} = \begin{bmatrix} x & s & \cdots & s \\ s & \cdots & \cdots & \vdots \\ \vdots & \cdots & \mathbf{B} & \vdots \\ s & \cdots & \cdots & \vdots \end{bmatrix}$$

B is said to be the *core of* **C** and the *s*'s are the *borders* of **B** in **C**. **C** is said to be in *bordered form*. The variables s and x can be realized in the cases described below.



■ *Fig. 1.* 2-level *Cretan* matrices of order 13 and 21: *a* — $\mathbf{CM}(13; 2; 9.60)$; *b* — $\mathbf{CM}(21; 2; 10)$

• Using Regular Hadamard Matrices

For details and constructions many of the known Regular Hadamard Matrices the interested reader is referred to [8, 7, 22].

Lemma 3. *Let* **M** *be a regular Hadamard matrix of order* $4m^2$ *with* $2m^2 + m$ *positive elements per row and column. Then forming* **C** *as follows*

$$\mathbf{C} = \begin{bmatrix} 1 & s & \cdots & s \\ s & \cdots & \cdots & \vdots \\ \vdots & \cdots & \frac{1}{2m}\mathbf{M} & \vdots \\ s & \cdots & \cdots & \vdots \end{bmatrix}$$

gives a *Cretan*$(4n^2 + 1;\ 4;\ 1)$ *matrix or* $\mathbf{CM}\left(4m^2+1;4;1;\left(0,1,\frac{1}{2m},\frac{-1}{2m}\right)\right)$.

Proof. For **C** to be a *Cretan* matrix it must satisfy the orthogonality, radius and characteristic equations. These are

$$\mathbf{CC}^{\mathrm{T}} = \left(1+4m^2s^2\right)\mathbf{I}_{4m^2+1} = \left(s^2+4m^2\right)\mathbf{I}_{4m^2+1} = \omega\mathbf{I}_{4m^2+1}$$

for the orthogonality equation, giving $s = 0$, $\omega = 1$ for the radius equation and 0 for the characteristic equations.

Hence we have a matrix of order $4m^2 + 1$ with elements 0, 1, $\pm\frac{1}{2m}$ satisfying the required *Cretan* equations.

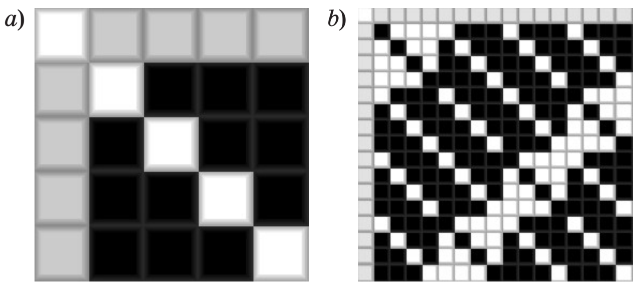
Corollary 1. *Since there exists a regular* (*symmetric*) *Hadamard matrix of order* $4 = 2^2$, $4^2 = 2^{2^2}$, $4^{4^2} = 2^{2^{2^2}}$, *..., there is a Cretan* $= \left(2^{2^{2^2}} \ldots + 1; 4; 1\right)$ *for n a Fermat number.*

Proof. Let **S** be the regular symmetric Hadamard matrix of order 4. Then the Kronecker product

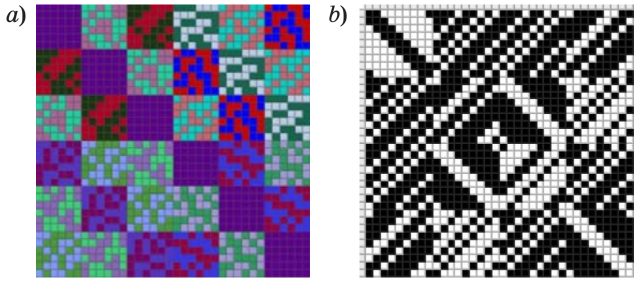
$$\mathbf{S} \times \mathbf{S} \times \ldots \times \mathbf{S}$$

is the required core for the construction in Lemma 3.

Example 3. Purported examples of pure Fermat matrices in Fig. 2, *a*, *b* for orders 5 and 17: levels *a*, *b* are white and black colours, the border level *s* is given in grey. However the reader is cautioned that though the figures appear to be *Cretan* matrices they are not. They are based on SBIBD, including the regular Hadamard matrix *SBIBD*$(4m^2, 2m \pm m, m \pm m)$ and require $c = a$. We note though that when $c = a \neq 1$ the radius and characteristic equations do not give meaningful real solutions.



■ ***Fig. 2.*** Orthogonal *Cretan*(Fermat) matrices for Fermat numbers 5 (*a*) and 17 (*b*)



■ ***Fig. 3.*** Regular Hadamard matrix of order 36 (*a*) and a 3-level *Cretan*(37) (*b*)

Example 4. See Fig. 3, *a*, *b* for examples of a regular Hadamard matrix of order 36 and a purported new Balonin — Seberry type of 3-level *Cretan*(37) with complex entries that is a orthogonal matrix of order 37. A real *Cretan*(37; 2) does exist from Theorem 2 above (see example).

Using Normalized Weighing Matrix Cores

The next construction is not valid in the real numbers. However we can allow *Cretan* matrices to have complex elements and choose the diagonal to be $i = \sqrt{-1}$.

Lemma 4. *Suppose there exists a normalized conference matrix*, **B**, *of order* $4t + 2$, *that is a* **W**$(4t + 2, 4t + 1)$. *Then* **B** *may be written as*

$$\mathbf{B} = \begin{bmatrix} i & 1 & \cdots & 1 \\ 1 & \cdots & \cdots & \vdots \\ \vdots & \cdots & \mathbf{F} & \vdots \\ 1 & \cdots & \cdots & \vdots \end{bmatrix}.$$

This is a Cretan matrix.

Removing the first row and column of **B** to study the core **F** is unproductive.

Generalized Hadamard Matrices and Generalized Weighing Matrices

We first note that the matrices we study here have elements from groups, abelian and non-abelian, (see [11–13, 23, 24] for more information) and may be written in additive or multiplicative notation. The matrices may have real elements, elements $\{1, -1\}$, elements $|n| \leq 1$, elements $\{1, i, i^2 = -1\}$, elements $\{1, i, -1, -i, i^2 = -1\}$, integer elements $\{a + ib, i^2 = -1\}$, n-th roots of unity, the quaternions $\{1$ and $i, j, k, i^2 = j^2 = k^2 = -1, ijk = -1\}$, $(a + ib) + j(c + id)$, a, b, c, d, *integer* and i, j, k quaternions or otherwise as specified.

We use the notations $\mathbf{B}^{\mathrm{T}}$ for the transpose of $\mathbf{G}$, $\mathbf{B}^{\mathrm{H}}$ for the group transpose, $\mathbf{B}^{\mathrm{C}}$ for the complex conjugate of $\mathbf{B}^{\mathrm{T}}$, $\mathbf{B}^{\mathrm{Q}}$ for the quaternion conjugate and $\mathbf{B}^{\mathrm{V}}$ for the quaternion conjugate transpose.

In all of these matrices the inner product of distinct rows a and b is $a - b$ or ab^{-1} depending on whether the group is written in additive or multiplicative form.

• **Generalized orthogonality:** A *generalized Hadamard matrix*, or *difference matrix*, $\mathbf{GH}(gn, g)$, of order $h = gn$, over a group of order g, has the inner product of distinct rows the whole group the same *number* of n times. The inner product is $\left\{g_{i1}g_{j1}^{-1}, \ldots, g_{ih}g_{jh}^{-1}\right\}$. For example

$$\mathbf{G} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & a & b & ab \\ 1 & b & ab & a \\ 1 & ab & a & b \end{bmatrix};\ \mathbf{GG}^{\mathrm{H}} = (\text{group})I_4 = (Z_2 \times Z_2)I$$

orthogonality is because of the definition of the inner product.

• **Butson Hadamard matrix** [11]

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix};\ \mathbf{BB}^{\mathrm{C}} = 3\mathbf{I}_3, \omega^3 = 1, 1 + \omega + \omega^2 = 0$$

is said to be a Butson Hadamard matrix. Orthogonality depends on the fact that the n nth roots of unity add to zero.

• **A generalized Hadamard matrix** [11, 12, 13], $\mathbf{GH}(np, G)$, where G is a group of order p, can also be written in additive form for example:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 2 & 1 & 0 & 1 & 2 \\ 0 & 2 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{bmatrix} \text{ is a } \mathbf{GH}(6, Z_3).$$

• **A generalized weighing matrix**, $\mathbf{W} = \mathbf{GW}(np, G, k)$ [23], where G is a group of order p, has w nonzero elements in each column and $\mathbf{W}$ is orthogonal over $\mathbf{G}$. The following two matrices are additive and multiplicative $\mathbf{GW}(5, Z_3)$, respectively:

$$\begin{bmatrix} * & 0 & 0 & 0 & 0 \\ 0 & * & 1 & 2 & 0 \\ 0 & 1 & * & 0 & 2 \\ 0 & 2 & 0 & * & 1 \\ 0 & 0 & 2 & 1 & * \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & \omega & \omega^2 & 1 \\ 1 & \omega & 0 & 1 & \omega^2 \\ 1 & \omega^2 & 1 & 0 & \omega \\ 1 & 1 & \omega^2 & \omega & 0 \end{bmatrix}.$$

* is zero but not the zero of the group.

Theorem 4. *Any generalized Hadamard matrix or generalized weighing matrix is a* $\mathbf{CM}(n; g)$ *over the group* $\mathbf{G}$, *of order* g, *which may be the roots of unity.*

The Kronecker Product of *Cretan* Matrices

Lemma 5. *Suppose* $\mathbf{A}$ *and* $\mathbf{B}$ *are* $\mathbf{CM}(n_1; \tau_1; \omega_1)$ *and* $\mathbf{CM}(n_2; \tau_2; \omega_2)$ *then the Kronecker product of* $\mathbf{A}$ *and* $\mathbf{B}$ *written* $\mathbf{A}{\times}\mathbf{B}$ *is a* $\mathbf{CM}(n_1n_2; \tau; \omega_1\omega_2)$ *where* τ *depends on* τ_1 *and* τ_2.

■ ***Table 1.*** Some *Cretan* $\mathbf{CM}(4t + 1)$, $3 \le 4t + 1 \le 199$

From Regular Hadamard Matrices ($\omega = 1$)	5 17 37 65 101 145 197				
From Difference Sets (ds)					
v	k	λ	Existence	Difference set	Comment
13	4	1	All known	PG(2, 3)	Unique Hall [28]
21	5	1	All known	PG(2, 4)	Unique Hall [28]
37	9	2	Exists	Biquadratic residue ds	Hall [28]
45	12	3	All known	—	La Jolla [20]
57	8	1	All known	PG(2, 7)	Unique Hall [28]
73	9	1	All known	PG(2, 8)	Unique Hall [28]
85	21	5	Exists	PG(3, 4)	[20]
101	25	6	Exists	Biquadratic residue ds	Hall [28]
109	28	7	Exists	Biquadratic residue ds	Hall [28]
121	40	13	Exists	PG(4, 3)	[20]
133	33	8	Exists	—	La Jolla [20]
197	49	12	Exists	Biquadratic residue ds	Hall [28]
Kronecker Product				All orders which are the product a known order and of prime power $\equiv 3 \pmod 4$	

Example 5. From [6, 25] we see that **CM**(3; 2; 2.25), **CM**(7; 2; 5.03) and **CM**(7; 2; 3.34) exist so there exist **CM**(21; 3; 11.32) and **CM**(21; 3; 7.52). The Hadamard — *Cretan* bound gives, for $n = 21$, radius ≤21.

From Balonin and Seberry [6] we have that since an $SBIBD\left(p^r, \frac{p^r-1}{2}, \frac{p^r-3}{4}\right)$ exists for all prime powers $p^r \equiv 3 \pmod 4$ there exist **CM**(p^r; 2; ω) for all these prime powers (see Proposition 2). Hence using Kronecker products in the previous theorem and writing n as a product of prime powers we have.

Theorem 5. *There exists a* **CM**(n; τ; ω) ω > 1 *for all odd orders* n, $n = \prod \rho \times p^{i_1} p^{i_2} \ldots$, *where* ρ *is an order for which a Cretan* **CM**(ρ = 4t + 1) *is known and* $p^{i_1} p^{i_2}, \ldots$ *are any prime powers* ≡3 (mod 4), *for some* τ *and* ω.

Table 1 gives the integers for which ρ is presently known. Similar theorems can be obtained for all even n.

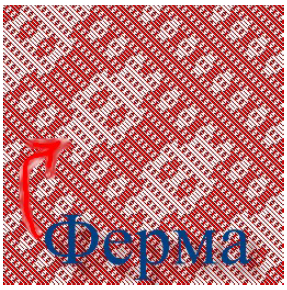
Remark 3. We note that τ depends on the actual construction used. Combining **CM**(n_1; 2; ω_1 : (a, b)) and **CM**(n_2; 2; ω_1 : (a, b)) gives **CM**($n_1 n_2$; 3; ω_{12} : (a^2, ab, b^2)). General formulae for τ from **CM** with different levels are left as an exercise.

The Difference between *Cretan*(4*t* + 1; τ) Matrices and Fermat Matrices

The first few pure Fermat numbers are $v = 3$, 5, 17, 257, 65 537, 4 294 967 297,... . We note these are all ≡1 (mod 4) and may be constructed using Corollary 1. Fig. 4 gives an early example of a Fermat matrix.

Finding 3-level orthogonal matrices of order ≡1 (mod 4) for non-pure Fermat numbers has proved challenging. Orders $n = 9$ and $n = 13$ are given in [4].

Orders $v = 2^{even} + 1$ called Fermat type matrices, pose an interesting class to study.



■ ***Fig. 4.*** Core of Russian Fermat Matrix from mathscinet.ru

Orders $4t + 1$, t is odd, are *Cretan*(4t + 1) — matrices; their order is neither a Fermat number ($2 + 1 = 3$, $2^2 + 1 = 4 + 1$, $2^{2^2} + 1 = 16 + 1$, $2^{2^{2^2}} + 1 = 256 + 1$, ...) nor a Fermat type number ($2^{even} + 1$). Examples of regular Hadamard matrices of order 36, giving the first **CM**(37; 3; 1) matrix of order 37 [3] where 37 is not a Fermat number or Fermat type number, have been placed at [26]. They use regular Hadamard matrices as a core and have the same, as any other Hadamard matrix, level functions. We call them *Cretan*(4t + 1) matrices and will consider them further in our future work.

Matrices of the *Cretan*(4t + 1) family made from Singer difference sets (see [21]) also have orders belonging to the set of numbers $4t + 1$, t odd: these are different from the three-level matrices of Balonin — Sergeev (Fermat) family [27, 19] with orders $4t + 1$, t is 1 or even.

Summary

In this paper we have given new constructions for **CM**(4t + 1). These are summarised in Table 1 for $4t + 1 < 200$. Table 2 gives 2-level and 3-level **CM**(4t ± 1).

■ ***Table 2.*** *Cretan* 2-level and 3-level **CM**(4t ± 1), $3 \le 4t + 1 \le 199$

v	Method	v	Method	v	Method
3	BM [4] + Prop. 2	5	BM [4]	7	BM [4] + Prop. 2
9	BM [4]	11	BM [4] + Prop. 2	13	BM [4]
15	Kronecker	17	—	19	Prop. 2
21	From SBIBD Table 1	23	Prop. 2	25	Kronecker
27	Prop. 2	29	—	31	Prop. 2
33	Kronecker	35	Kronecker	37	—
39	Kronecker	41	—	43	Prop. 2
45	From SBIBD Table 1	47	Prop. 2	49	Kronecker
51	—	53	—	55	Kronecker
57	From SBIBD Table 1	59	Prop. 2	61	—

■ ***Finish of table 2***

v	Method	*v*	Method	*v*	Method
63	Kronecker	65	Kronecker	67	Prop. 2
69	Kronecker	71	Prop. 2	73	From SBIBD Table 1
75	Kronecker	77	Kronecker	79	Prop. 2
81	Prop. 2	83	—	85	From SBIBD Table 1
87	—	89	—	91	Kronecker
93	Kronecker	95	Kronecker	97	—
99	Kronecker	101	From SBIBD Table 1	103	Prop. 2
105	Kronecker	107	Prop. 2	109	From SBIBD Table 1
111	—	113	—	115	Kronecker
117	Kronecker	119	—	121	From SBIBD Table 1
123	—	125	Kronecker	127	Prop. 2
129	Kronecker	131	Prop. 2	133	From SBIBD Table 1
135	Kronecker	137	—	139	Prop. 2
141	Kronecker	143	—	145	—
147	Kronecker	149	—	151	Prop. 2
153	—	155	Kronecker	157	—
159	—	161	Kronecker	163	Prop. 2
165	Kronecker	167	Prop. 2	169	Kronecker
171	Prop. 2	173	—	175	Kronecker
177	Kronecker	179	Prop. 2	181	—
183	—	185	—	187	—
189	Kronecker	191	Prop. 2	193	—
195	Prop. 2	197	From SBIBD Table 1	199	Prop. 2

Conclusions

Cretan matrices are a very new area of study. They have many research lines open: what is the minimum number of variables that can be used; what are the determinants and radii that can be found for *Cretan*(n; τ) matrices; why do the congruence classes of the orders make such a difference to the proliferation of *Cretan* matrices for a given order; find the *Cretan* matrix with maximum and minimum determinant for a given order; can one be found with fewer levels?

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Критские матрицы порядков $4t + 1$

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Цель: дать критские матрицы $Cretan(4t + 1)$ порядков $4t + 1$ — ортогональные матрицы с элементами, ограниченными по модулю ≤1 (ранее публиковались критские матрицы типа $Cretan(4t + 1)$ определенных порядков 5, 9, 13, 17 и 37). **Результаты:** приведено неограниченно много новых критских матриц $Cretan(4t + 1)$, конструируемых при помощи регулярных матриц Адамара, симметричного сбалансированного блочного дизайна SBIBD$(4t + 1; k; \lambda)$, взвешенных матриц, обобщенных матриц Адамара и произведения Кронекера. Предложено неравенство для радиуса матриц и дана конструкция критской матрицы для каждого порядка $n \geq 5$. **Практическая значимость:** критские матрицы $Cretan(4t + 1)$ имеют непосредственное практическое применение к проблемам помехоустойчивого кодирования, сжатия и маскирования видеоинформации.

Ключевые слова — матрицы Адамара, регулярные матрицы Адамара, ортогональные матрицы, симметричный сбалансированный блочный дизайн (SBIBD), критские матрицы, взвешенные матрицы, обобщенные матрицы Адамара, 05B20.